APPENDIX TO PAPER 5

THE MATHEMATICAL RELATIONSHIP BETWEEN EUCLIDEAN SPACE AND RAPIDITY SPACE.

(Paper 5: *Velocity addition formula from euclidean space analogue*, www.euclideanrelativity.net , S. Kanagaraj, 20 Dec 2009).

□ The relationships between velocity υ, rapidity α, Lorentz ratio γ, proper velocity V, orientation angle ϕ and spacetime angle θ are shown below.

1. Table 1: The relationships between υ, α, γ, V, ϕ and θ.

	$f(\upsilon)$	$f(\alpha)$	$f(\gamma)$	$f(V)$	$f(\phi)$	$f(\theta)$
Velocity, υ	υ	$\tanh \alpha$	$\frac{\sqrt{\gamma^2-1}}{\gamma}$	$\frac{V}{\sqrt{1+V^2}}$	$\sin \phi$	$\cos \theta$
Rapidity, α	$\text{arctanh}\, \upsilon$	α	$\text{arctanh} \frac{\sqrt{\gamma^2-1}}{\gamma}$	$\text{arcsinh}\, V$	$\text{arctanh}(\sin\phi)$	$\text{arctanh}(\cos \theta)$
Lorentz ratio, γ	$\frac{1}{\sqrt{1-\upsilon^2}}$	$\cosh \alpha$	γ	$\sqrt{1+V^2}$	$\sec \phi$	$\text{cosec}\, \theta$
Proper velocity, V	$\frac{\upsilon}{\sqrt{1-\upsilon^2}}$	$\sinh \alpha$	$\sqrt{\gamma^2-1}$	V	$\tan \phi$	$\cot \theta$
Orientation angle, ϕ	$\arcsin \upsilon$	$\arcsin(\tanh\alpha)$	$\text{arcsec}\, \gamma$	$\arctan V$	ϕ	$\arcsin(\cos \theta)$
Spacetime angle, θ	$\arccos \upsilon$	$\arccos(\tanh\alpha)$	$\text{arccosec}\, \gamma$	$\text{arccot}\, V$	$\arccos(\sin \phi)$	θ

Note: The relationship between υ, α and γ are familiar while the variables V, ϕ and θ are introduced in euclidean interpretation of relativity.

2. The differential relationship between υ, α, γ, V, ϕ and θ.

(i) From $\upsilon = tanh\ \alpha$, $\dfrac{d\upsilon}{d\alpha} = sech^2\ \alpha$ and $\dfrac{d\alpha}{d\upsilon} = cosh^2\ \alpha$

From Table 1, $cosh\ \alpha = sec\ \phi = cosec\ \theta = \dfrac{1}{\sqrt{1-\upsilon^2}} = \gamma = \sqrt{1+V^2}$

Thus $\dfrac{d\alpha}{d\upsilon} = sec^2\ \phi = cosec^2\ \theta = cosh^2\ \alpha = \dfrac{1}{1-\upsilon^2} = \gamma^2 = (1+V^2)$

(ii) From $\upsilon = sin\ \phi$, $\dfrac{d\upsilon}{d\phi} = cos\ \phi$ and $\dfrac{d\phi}{d\upsilon} = sec\ \phi$

From $\upsilon = cos\ \theta$, $\dfrac{d\upsilon}{d\theta} = -\,sin\ \theta$ and $\dfrac{d\theta}{d\upsilon} = -\,cosec\ \theta$

Thus $\dfrac{d\phi}{d\upsilon} = sec\ \phi = -\dfrac{d\theta}{d\upsilon} = cosec\ \theta = cosh\ \alpha = \dfrac{1}{\sqrt{1-\upsilon^2}} = \gamma = \sqrt{1+V^2}$

(iii) From $V = \dfrac{\upsilon}{\sqrt{1-\upsilon^2}} = \upsilon(1-\upsilon^2)^{-1/2}$

$$\frac{dV}{d\upsilon} = \upsilon\{[-\frac{1}{2}(1-\upsilon^2)^{-3/2}](-2\upsilon)\} + (1-\upsilon^2)^{-1/2}$$

$$= \frac{\upsilon^2}{(1-\upsilon^2)\sqrt{1-\upsilon^2}} + \frac{1}{\sqrt{1-\upsilon^2}}$$

$$= \frac{\upsilon^2 + (1-\upsilon^2)}{(1-\upsilon^2)\sqrt{1-\upsilon^2}}$$

$$= \left(\frac{1}{\sqrt{1-\upsilon^2}}\right)^3$$

$$= \gamma^3$$

Thus $\dfrac{dV}{d\upsilon} = \gamma^3 = sec^3\ \phi = cosec^3\ \theta = cosh^3\ \alpha$

(iv) $\dfrac{d\phi}{dV} = \dfrac{d\phi}{d\upsilon} \times \dfrac{d\upsilon}{dV}$

$= \gamma \times \dfrac{1}{\gamma^3}$

$= 1/\gamma^2$

Thus $\dfrac{dV}{d\phi} = \gamma^2 = sec^2\,\phi = -\dfrac{dV}{d\theta} = cosec^2\theta = cosh^2\,\alpha = \dfrac{1}{1-\upsilon^2} = 1+V^2 = \dfrac{d\alpha}{d\upsilon}$

(v) $\dfrac{dV}{d\alpha} = \dfrac{dV}{d\upsilon} \times \dfrac{d\upsilon}{d\alpha}$

$= \gamma^3 \times \dfrac{1}{\gamma^2}$

$= \gamma$

Thus $\dfrac{dV}{d\alpha} = \gamma = sec\,\phi = \dfrac{1}{\sqrt{1-\upsilon^2}} = cosh\,\alpha = \sqrt{1+V^2} = \dfrac{d\phi}{d\upsilon} = -\dfrac{d\theta}{d\upsilon} = cosec\,\theta$

(vi) $\dfrac{d\alpha}{d\phi} = \dfrac{d\alpha}{d\upsilon} \times \dfrac{d\upsilon}{d\phi}$

$= \gamma^2 \times \dfrac{1}{\gamma}$

$= \gamma$

<u>Summary of the differential relationship</u>

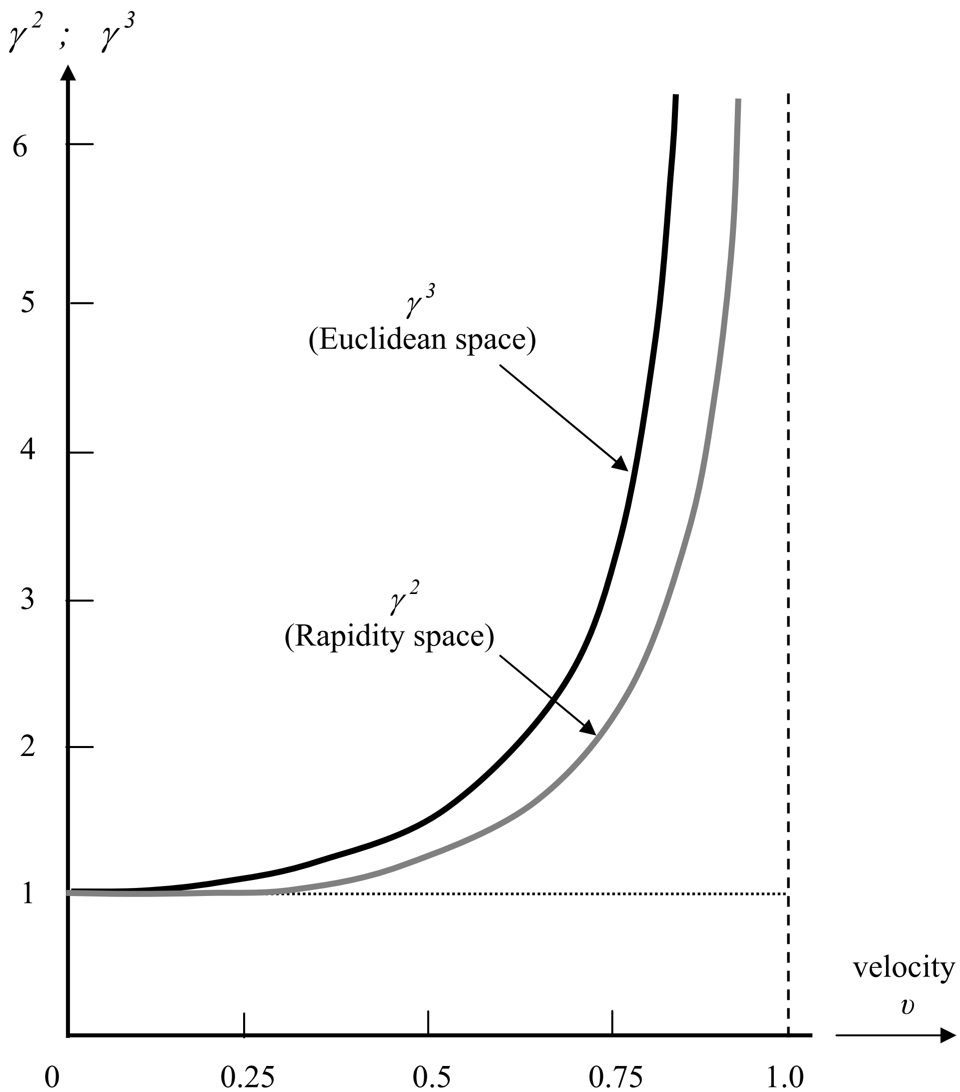
$$-\frac{d\theta}{d\upsilon} = \frac{d\phi}{d\upsilon} = \frac{dV}{d\alpha} = \frac{d\alpha}{d\phi} = \gamma$$

$$-\frac{dV}{d\theta} = \frac{d\alpha}{d\upsilon} = \frac{dV}{d\phi} = \gamma^2$$

$$\frac{dV}{d\upsilon} = \gamma^3$$

3. Investigating the addition of rapidity α and proper velocity V from the integral functions of rapidity space and euclidean space.

In rapidity space, $\frac{d\alpha}{dv} = \gamma^2$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$. Thus $\alpha = \int \gamma^2 \, dv$. This translates as rapidity α is the area under the graph of γ^2 vs v. In euclidean space, $\frac{dV}{dv} = \gamma^3$, thus $V = \int \gamma^3 \, dv$. This translates as the proper velocity V, is the area under the graph of γ^3 vs v.



Graph of γ^2 vs v and γ^3 vs v for range of $0 < v < 1$

For the velocity v range $0 < v < 1$, both the graphs γ^2 & γ^3 ranges from 1 to ∞ and at low velocities both remains close to 1, showing $v \approx \alpha \approx V$ when $v << 1$. Since $\gamma^3 > \gamma^2$, for the same area under the graphs representing the proper velocity V and rapidity α, the velocity v from euclidean space is slightly less than its corresponding value in rapidity space. (This is consistent with *Fig 2* graph in paper 5).

Noting that $\gamma = sec\ \phi$ and $d\upsilon = cos\ \phi\ d\phi$ (from *2i* & *2ii* above), the integral function of rapidity α and proper velocity V can be expressed in terms of ϕ instead of υ.

From $\alpha = \int_0^{\upsilon} \gamma^2\ d\upsilon$, substituting γ and $d\upsilon$,

$$= \int_0^{\phi} sec^2\phi\ cos\ \phi\ d\phi$$

$$= \int_0^{\phi} sec\ \phi\ d\phi$$

$$= ln(sec\ \phi + tan\ \phi)$$

Thus $e^{\alpha} = sec\ \phi + tan\ \phi$. From this, $e^{\alpha} = \dfrac{1 + sin\ \phi}{cos\phi}$ and $e^{-\alpha} = \dfrac{cos\ \phi}{1 + sin\phi}$. Substituting into $\dfrac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}$, the relationship simplifies as $tanh\ \alpha = sin\ \phi$, consistent with as in Table 1.

From $V = \int_0^{\upsilon} \gamma^3\ d\upsilon$

$$= \int_0^{\phi} sec^3\phi\ cos\ \phi\ d\phi$$

$$= \int_0^{\phi} sec^2\phi\ d\phi$$

$$= tan\ \phi$$

This is consistent with $V = tan\ \phi$ as in Table 1.

Since $sec\ \phi = \gamma$, for the graph γ vs ϕ (rapidity space) and γ^2 vs ϕ (euclidean space), the ϕ range is $0 < \phi < \frac{\pi}{2}$ corresponding with υ range $0 < \upsilon < 1$. Alternatively expressed as a $f(\theta)$, the resulting equations are consistent with the above.

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