Relativistic Speed Composition Formulas

Someone was asking about how we can be sure the correct relativistic speed composition formula is $(u+v)/(1+uv)$ rather than, say

$$\sin\{ \arctan[\tan(\arcsin(u)) + \tan(\arcsin(v))] \}$$

I'm not aware of any particular motivation at the present time to seek a new addition formula for speeds, at least not on the macroscopic scale, because the relativistic rule works splendidly, and is the only rule consistent with the overall relativistic structure that has been so successful at describing and predicting physical phenomena. On the other hand, it's sometimes interesting to review the simple algebraic equations associated with relativity and compare them - from a purely formalistic standpoint - with other functions of the same general class, to clarify what distinguishes the formulae that work from those that don't.

Letting v12, v23, and v13 denote the pairwise velocities (in geometric units) between three co-linear particles P1, P2, P3, a composition formula relating these speeds can generally be expressed in the form

$$f(v13) = f(v12) + f(v23)$$

where f is some function that transforms speeds into a domain where they are simply additive. It's clear that f must be an "odd" function, i.e., $f(-x) = -f(x)$, to ensure that the same composition formula works for both positive and negative speeds. This rules out transforms such as $f(x) = x^2$, $f(x) = \cos(x)$, and all other "even" functions.

The general "odd" function expressed as a power series is a linear combination of odd powers, i.e.,

$$f(x) = c1\, x + c3\, x^3 + c5\, x^5 + c7\, x^7 + \ldots$$

so we can express any such function in terms of the coefficients [c1,c3,...]. For example, if we take the coefficients [1,0,0,...] we have the simple transform $f(x) = x$, which gives the Galilean composition formula

$$v13 = v12 + v23 \qquad [1]$$

For another example, suppose we "weight" each term in inverse proportion to the exponent by using the coefficients [1, 1/3, 1/5, 1/7,...]. This gives the transform

$$f(x) = x + x^3/3 + x^5/5 + \ldots = \operatorname{atanh}(x)$$

leading to Einstein's relativistic composition formula

$$\operatorname{atanh}(v13) = \operatorname{atanh}(v12) + \operatorname{atanh}(v23) \qquad [2a]$$

From the identity $\operatorname{atanh}(x) = \ln[(1+x)/(1-x)]/2$ we also have the equivalent multiplicative form

$$\left(\frac{1 + v13}{1 - v13}\right) = \left(\frac{1 + v12}{1 - v12}\right)\left(\frac{1 + v23}{1 - v23}\right) \qquad [2b]$$

which is arguably the most natural form of the relativistic speed composition law. In fact the velocity parameter $p = (1+v)/(1-v)$ gives very natural expressions for many other observables as well, including relativistic doppler shift = sqrt(p)

spacetime interval between two inertial particles each 1 unit of proper time past their point of intersection = p^(1/4) - p^(-1/4)

Incidentally, to give an equilateral triangle in spacetime, this last equation shows that two particles must have a mutual speed of sqrt(5)/3 = 0.745...

Anyway, pressing on with this (admittedly superficial) approach to divining the correct speed composition law on purely formalistic grounds, consider the set of (odd) coefficients [1,1,1,...], corresponding to the "odd geometric series"

$$f(x) = x + x^3 + x^5 + x^7 + \ldots = x/(1 - x^2)$$

If we adopt this transform, our composition formula would be

$$\frac{v13}{1 - v13^2} = \frac{v12}{1 - v12^2} + \frac{v23}{1 - v23^2} \qquad [3]$$

Here we see that each speed is normalized by the square of the relativistic "gamma" factor. A variation on this would be to correct each speed with gamma itself, i.e.,

$$\frac{v13}{\text{sqrt}[1 - v13^2]} = \frac{v12}{\text{sqrt}[1 - v12^2]} + \frac{v23}{\text{sqrt}[1 - v23^2]} \qquad [4a]$$

In trigonometric form this can be written as

$$\tan(\text{asin}(v13)) = \tan(\text{asin}(v12)) + \tan(\text{asin}(v23)) \qquad [4b]$$

and using some basic trig identities it can also be expressed in terms of hyperbolic functions as

$$\sinh(\text{atanh}(v13)) = \sinh(\text{atanh}(v12)) + \sinh(\text{atanh}(v23)) \qquad [4c]$$

Notice that [4c] is related to the relativistic formula [2a] simply by applying the hyperbolic sine to each term. The power series coefficients of sinh(atanh(x)) are [1, 1/2, 3/8, 5/16,...], which seem somewhat less "natural" than any of the previous coefficient sets. Furthermore, both [3] and [4] suffer from the fact that although they are singular at arguments of 1, the slope of v13 does not go to zero as v12 and v23 approach 1.

To remedy this we could instead apply the INVERSE hyperbolic sine to the terms of [2], giving the composition formula

$$\text{asinh}(\text{atanh}(v13)) = \text{asinh}(\text{atanh}(v12)) + \text{asinh}(\text{atanh}(v23)) \qquad [5]$$

which DOES have a zero slope at arguments of 1. Not surprisingly, the discrepancy between [5] and the relativistic formula [2] is about the same as the discrepancy between [4] and [2], but in the opposite direction. As a result, the average of the compositions based on [4] and [5] is nearly indistinguishable from the relativistic composition [2].