Momentum and energy equation from Euclidean space analogue.

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Abstract
The momentum and energy variation equations are derived from a 4-Euclidean space analogue model of a moving body. The equations are expressed in terms of a real angular rotation ϕ (or alternatively θ) in trigonometric form. The relativistic variations are explained in terms of a geometric property without invoking the problematic relativistic mass concept.

1.0 Introduction.

The relativistic variation in space and time were derived from the Euclidean space analogue of a moving body, the Euclidean space (ES) diagram [1]. In this paper, the ES diagram is applied to derive the momentum and energy variation equation. The variations are explained as due to a real 4-dimensional rotation that uniquely corresponds to the velocity v of a body and expressed in trigonometric form.

2.0 Momentum equation.

The momentum p of a body is by definition the product of its mass with velocity. Thus

$$momentum = mass \times velocity \, .$$

In the ES diagram, the velocity of a body in 4-Euclidean space analogue is the proper velocity V. We adopt that mass m of a body is a constant.

Thus, $p = mV$ ……. *Eqn 1*

Eqn 1 corresponds to classical mechanics with momentum p directly proportional to velocity V.

In the ES diagram, the proper velocity V is the rate of the proper displacement, S, thus $V = \frac{dS}{dt}$. Its observational viewpoint component, the conventional velocity, υ, is the rate of observed displacement, s, thus $\upsilon = \frac{ds}{dt}$. The path of the proper displacement is inclined at an angle ϕ from the path of the observed displacement.

Since $V = \frac{\upsilon}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ and Lorentz ratio $\gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$, thus $V = \upsilon\gamma$

When $\upsilon \ll c$, $\gamma \approx 1$ and $V \approx \upsilon$. For this special case, the momentum equation reduces to a good approximation as $p = m\upsilon$. Noting that $\sin\phi = \upsilon/c$, the momentum equation $p = mV$ expressed as a function of the orientation angle ϕ is

$$p = mc \tan\phi. \;\ldots.\; Eqn\ 2\ .$$

With mc a constant, from *Eqn 2*, the momentum variation is due to a rotation ϕ in 4-Euclidean space. When $\upsilon \ll c$, ϕ is small. For this case $\tan\phi \approx \sin\phi$, thus $p = mc\tan\phi \approx mc\sin\phi = m\upsilon$. Alternatively expressed as a function of the spacetime angle θ, $p = mV = mc\cot\theta$. For the case $\upsilon \ll c$, $\theta \approx \pi/2$ and $\cot\theta \approx \cos\theta$ and consequently $p = mc\cot\theta \approx mc\cos\theta = m\upsilon$. The required positive (+*ve*) and negative (–*ve*) sign change for receding and approaching motion appears naturally for both the angles ϕ and θ .

3.0 Acceleration

For a frame physically moving at increasing velocity υ along the x_1-axis, there is a corresponding increase in its V and ϕ values. For this case where the velocity varies for different time intervals, the instantaneous conventional velocity υ, is $\upsilon = \frac{\delta s}{\delta t}$ where δs is the incremental displacement along the x_1-axis and δt is the incremental time interval. In the limit $\delta t \rightarrow 0$, $\upsilon = \frac{ds}{dt}$. Similarly, its instantaneous proper velocity V along the inclined x_1'- axis is $V = \frac{\delta S}{\delta t}$ where δS is its incremental displacement along the inclined x_1'- axis and δt is the incremental time interval. In the limit $\delta t \rightarrow 0$, $V = \frac{dS}{dt}$.

Conventionally, the acceleration , a , of a frame is the rate of change in its (conventional) velocity υ , thus $a = \frac{d\upsilon}{dt}$. The proper acceleration A of a frame is the rate of change in its proper velocity V , thus $A = \frac{dV}{dt}$. Similar to how we denoted the velocity of a body as υ and V , we denote the acceleration of a body as the conventional acceleration, a , and the proper acceleration, A. [1]

These relationships are summarized as

$$a = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

$$A = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2 S}{dt^2}$$

A re-expressed as a function of υ is :-

$$A = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{\upsilon}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\right) = \frac{d}{dt}(\upsilon\gamma)$$

When $\upsilon \ll c$, $\gamma \approx 1$ and $A \approx \frac{d\upsilon}{dt} = a$. Also since $\gamma = sec\ \phi = cosec\ \theta$ [1], for this case $\phi \approx 0$, $sec\ \phi \approx 1$ and $\theta \approx \pi/2$, $cosec\ \theta \approx 1$ which is consistent with the above.

4.0 Force relationship

The force applied on a body and the rate of change in its velocity is

Force = (*Mass of body*) × (*The rate of change in its velocity*) .

[1] Applying the ES diagram, the displacement s, velocity υ and acceleration a are respectively transformed to its 'proper' displacement S, 'proper' velocity V and 'proper' acceleration A resulting in a proper quantities formulation of relativistic dynamics.

It is assumed the body is rigid and moving freely without changing its mass. Classically, this force equation in its vector differential form is

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = m\frac{d^2\boldsymbol{s}}{dt^2}$$

where

$\boldsymbol{F}$ = force applied (along the direction of its motion)
$\boldsymbol{a}$ = conventional (i.e. observed) acceleration
$\boldsymbol{v}$ = conventional (i.e. observed) velocity
$\boldsymbol{s}$ = observed displacement
t = time (using reference observer's clock)
m = mass of body

Since m is a constant, $\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{d\boldsymbol{p}}{dt}$

These classical equations were formulated on the assumption a body moves in Euclidean space. The rate of change in velocity v is proportional to the force F consistent with Newton's 2nd law of motion. It implies that if the force is applied continuously, the velocity would increase indefinitely without any limitation. With the introduction of relativistic physics, it became evident the assumption that a body moves in Euclidean space is only valid to a good approximation for the case $v << c$ and that v reaches a limitation of c. This implies to apply the classical equations with mathematical validity, a moving body should be referenced to a Euclidean space analogue. The ES diagram satisfies this requirement to study relativistic dynamics conveniently in close correspondence to classical physics.

We adopt the force equation in its vector differential form as

$$\boldsymbol{F} = m\boldsymbol{A} = m\frac{d\boldsymbol{V}}{dt} = m\frac{d^2\boldsymbol{S}}{dt^2} \quad \text{........... } Eqn\ 3$$

Where $\boldsymbol{A}$ = proper acceleration ; $\boldsymbol{V}$ = proper velocity ; $\boldsymbol{S}$ = proper displacement

Since m = constant, re-arranging, $\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{V}) = \frac{d\boldsymbol{p}}{dt}$

From *Fig 1*, for a force F along the x_1-axis, its component along the x_1'-axis that contributes to a change in the proper velocity V of the body in the ES diagram is $F\cos\phi$.

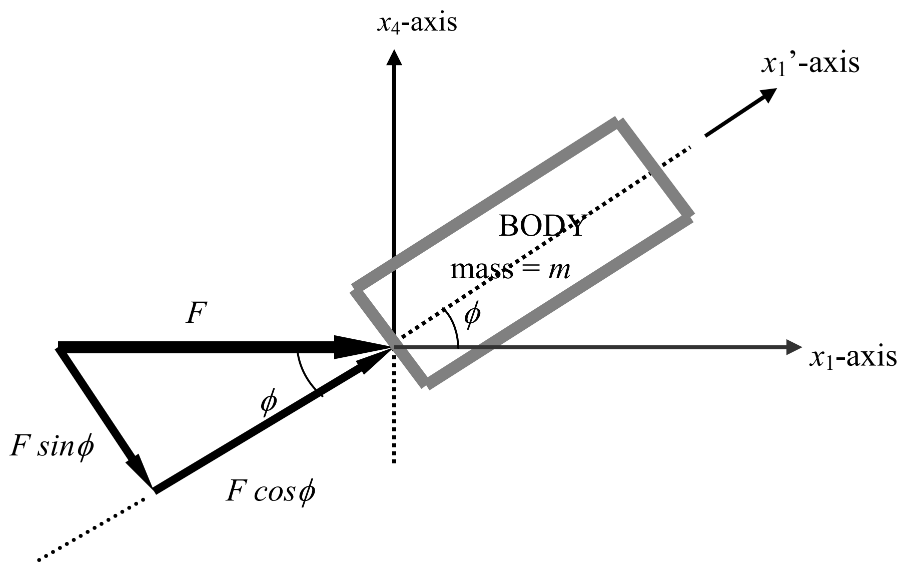

Figure 1 : The components of force *F* in relation to the ES diagram.

Since $\cos\phi = \sqrt{1-\frac{\upsilon^2}{c^2}}$, as υ increases, $\cos\phi$ decreases and the component force reduces with increasing speed. It would seem that more force F has to be applied to produce the same rate of increase of its velocity υ at higher speeds. As velocity υ of a body approaches c, ϕ approaches $\pi/2$ and $\cos\phi \to 0$. The component force along the inclined x_1'-axis also approaches 0 implying the limitation of υ relative to an observer is c consistent with observations.

5.0 Energy equation

The work-energy theorem states the change in a body's kinetic energy is equal to the net work done on that body. We assume the investigated body is rigid and moving freely. The work done on the body, W, due to a force, F , along its direction of motion completely contributes towards changing the kinetic energy, E_k , of that body. For this case, $E_k = W$ and thus the incremental changes δE_k and δW are equal.

$$\delta E_k = \delta W \quad \ldots\ldots\ Eqn\ 4$$

Since work done (W) = force (F) × displacement (x), the incremental relationship is $\delta W = F\,\delta x$ [2]

If the force applied is inclined at an angle Φ from the direction of motion, this equation is written as

$$\delta W = F\cos\Phi\ \delta x \quad \ldots\ldots \quad Eqn\ 5$$

From *Eqn 4* & 5, $\delta E_k = F\cos\Phi\ \delta x$. In the limit as $\delta x \to 0$,

$$dE_k = F\cos\Phi\ dx \quad \ldots\ldots \quad Eqn\ 6$$

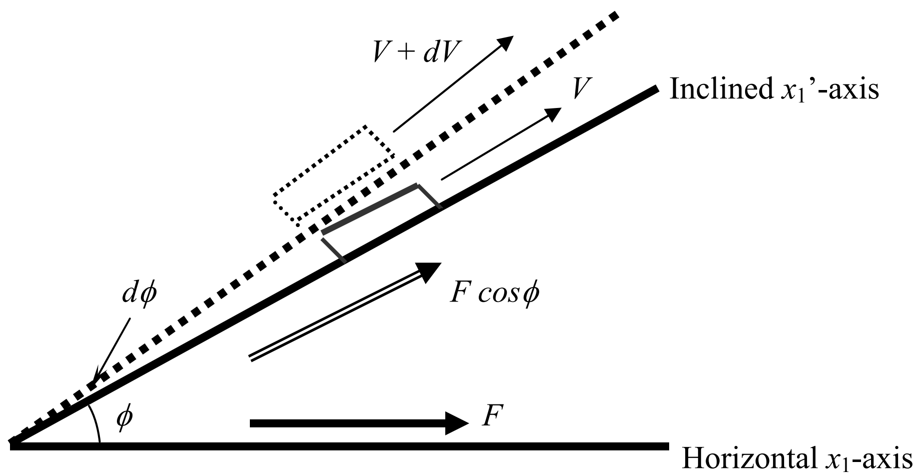

Figure 2 :The incremental change in *V* and ϕ due to a force *F*.

Fig 2 shows how the force F applied to a body along the x_1-axis (its observed direction of motion), is represented in the ES diagram where a body is studied analogous to moving at proper velocity V at an angle ϕ in 4- Euclidean space along the inclined x_1'-axis. dV and $d\phi$ are the body's incremental changes in proper velocity and orientation angle due to the force F.

In the ES diagram, the displacement S of a moving body is at an angle ϕ along the inclined x_1'-axis. Substituting the incremental displacement dx and angle Φ in *Eqn 6* with dS and ϕ respectively,

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[2] Strictly $\delta W \approx F\ \delta x$. However since the incremental variation in force δF is negligible as compared to F during the incremental displacement δx , the equation is justifiably expressed as $\delta W = F\ \delta x$.

$$dE_k = F\ cos\phi\ dS \ \ .... Eqn\ 7$$

Substituting $F = m\dfrac{dV}{dt}$ from *Eqn 3* into *Eqn 7*,

$$dE_k = m\left(\frac{dV}{dt}\right) cos\,\phi\ \ dS \ \ .... Eqn\ 8$$

Re-arranging *Eqn 8*, $dE_k = m\left(\dfrac{dS}{dt}\right) cos\,\phi\ \ dV$

Since $\dfrac{dS}{dt} = V$, $dE_k = mV\ cos\,\phi\ dV \ \ .... Eqn\ 9$

From the ES diagram, $V = c\ tan\,\phi$ and $\dfrac{dV}{d\phi} = c\ sec^2\phi$. Substituting V and dV into *Eqn 9* ,

$$dE_k = m(c\ tan\,\phi)\ cos\,\phi\ (c\ sec^2\phi)\ d\phi$$

$= mc^2\ tan\,\phi\ \ sec\,\phi\ \ d\phi \ \ .... Eqn\ 10$, where m and c are constants

Integrating both sides of *Eqn 10* ,

$$E_k = mc^2\int_0^{\phi} tan\,\phi\ sec\phi\ d\phi$$

$$= mc^2\ [sec\ \phi]_0^{\phi}$$

$$= mc^2(sec\,\phi - 1) \ \ .... Eqn\ \ 11$$

Expanding and re-arranging *Eqn 11* ,

$$mc^2\ sec\,\phi = E_k + mc^2 \ \ .... Eqn\ 11a$$

where E_k = the energy of the body due to its motion and mc^2 = a constant

If we define the total energy E of a body as the sum of its energy due to its motion, E_k, and a constant component when at rest, mc^2, from *Eqn 11a* we deduce that

$$E = mc^2 \sec\phi \quad \ldots\ldots \; Eqn\ 12$$

Eqn 11a and *12* implies that a rest body possesses energy equal to mc^2 consistent with the mass-energy equivalence. The energy variation is explained as due to a rotation ϕ in 4-Euclidean space in similarity with the momentum equation. Since $\sec\phi = [1 - \upsilon^2/c^2]^{-1/2} = \gamma = \operatorname{cosec}\theta$, the energy equation can alternatively be expressed as a function of the Lorentz factor γ or spacetime angle θ as $E = mc^2\gamma = mc^2 \operatorname{cosec}\theta$. As υ approaches c, ϕ approaches $\pi/2$ and θ approaches 0 and correspondingly E approaches infinity. Thus the velocity υ of a body can only approach but never reach c relative to an observer.

From binomial expansion,

$$E = mc^2 \left(1 + \frac{\upsilon^2}{2c^2} + \frac{3\upsilon^4}{8c^4} + \frac{5\upsilon^6}{16c^4} + \ldots\ldots \right)$$

Neglecting the higher orders from power 4 for the case $\upsilon << c$, the kinetic energy reduces to the classical equation.

6.0 Momentum-energy equation

Re-arranging equations *2* and *12*, $\tan\phi = \dfrac{p}{mc}$ and $\sec\phi = \dfrac{E}{mc^2}$

Substituting $\tan\phi$ and $\sec\phi$ into the trigonometric identity $\sec^2\phi = 1 + \tan^2\phi$,

$$\left(\frac{E}{mc^2}\right)^2 = 1 + \left(\frac{p}{mc}\right)^2$$

Re-arranging and simplifying,

$$E^2 = p^2c^2 + m^2c^4 \quad \ldots\ldots\ldots \; Eqn\ 13$$

The momentum-energy equation (*Eqn 13*) can alternatively be derived from the energy and momentum equations expressed as a $f(\theta)$ using the identity, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

Based upon the mathematical postulate, *sin* $\phi = \upsilon/c$, *Majernik* [2] has given the above expressions for *p*, *E* and *p-E* as a function of ϕ in trigonometric form.

7.0 Resolves relativistic mass problem.

In the above derivation, the kinematics of the momentum and energy variations are explained in terms of a geometric property as due to a rotation ϕ of a body in 4-Euclidean space without invoking the problematic relativistic mass concept. *Adler* [3] cites 3 sources [4],[5] & [6] arguing that the variations be explained in terms of geometric property and not relativistic mass. If the relativistic mass concept is invoked, then the relativistic mass $M = \dfrac{m}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$ and the momentum equation is written as $p = M\upsilon$. The force $F = \dfrac{dp}{dt} = \dfrac{d}{dt}(M\upsilon)$. Substituting *F* in the work-energy relationship, $dE_k = F\, ds$, then $E_k = \int_m^M \dfrac{d(M\upsilon)}{dt}\, ds$. Solving this, the energy equation is written as $E=Mc^2$. Since $M=m\gamma$, $p=m\gamma\upsilon$ and $E=m\gamma c^2$ which are consistent with our derivation. Our derivation based on constancy of mass corresponds with *Einstein*'s [4] view, *"It is not good to introduce the concept of the mass* $M = \dfrac{m}{\left(1-\upsilon^2/c^2\right)^{1/2}}$ *of a body for which no clear definition can be given. It is better to introduce no other mass than the 'rest mass' m "*.

8.0 Conclusion

The momentum *p* and energy *E* variation equations and the *p-E* relationship were derived from the ES diagram in terms of a real rotation ϕ in trigonometric form. The derived equations were consistent with the standard equations. The ES diagram which is a 4-Euclidean space analogue of a moving body offers as a viable alternative to study relativistic dynamics conveniently in close correspondence to classical physics.

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