VELOCITY ADDITION FORMULA FROM EUCLIDEAN SPACE ANALOGUE.

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ABSTRACT

By transforming the velocity of a frame analogous as moving in 4-euclidean space, dynamics is studied in close correspondence with classical physics in the euclidean space diagram. Subsequently the velocity addition formula is deduced as the arithmetical addition of proper velocities V. In the standard formulation it is the rapidities α from rapidity space which are arithmetically added. As a result, the velocity transformation graph of the formula slightly deviates from the standard one. An analysis reveals the characteristics of both the euclidean space and rapidity space graphs are identical. The formula fulfills the conditions to satisfy relativistic requirements in similarity with the standard one. The interpretation problems in concluding the velocity addition as a tested formula is reviewed.

1. The standard velocity addition formula reviewed.

In studying 3 collinear inertial frames *1*, *2* and *3*, observers in frame *1* and *2* observe frame *2* and *3* moving at velocity υ_1 and υ_2 along the same path respectively. From Lorentz transformation equations, the space-time coordinates for observations from frame *2* is transformed in terms of reference frame *1*. Subsequently, the velocity υ_3 of frame *3* relative to *1*, the velocity addition formula is derived (where *1* unit of velocity = *c*) as

$$\upsilon_3 = \frac{\upsilon_1 + \upsilon_2}{1 + \upsilon_1 \upsilon_2} \quad \text{........ } Eqn\ 1$$

In the space-time diagram, the relationship between velocity υ and the complex rotation Φ is $\tan \Phi = i\upsilon$ (where $i = \sqrt{-1}$). Thus $\tan \Phi_1 = i\upsilon_1$; $\tan \Phi_2 = i\upsilon_2$ and $\tan \Phi_3 = i\upsilon_3$ with the angular relationship as $\tan \Phi_3 = \tan (\Phi_1 + \Phi_2)$. Expanding and substituting the terms gives the same standard formula.

Comparing *Eqn 1* with the hyperbolic identity, $\tanh(\alpha_1 + \alpha_2) = \dfrac{\tanh \alpha_1 + \tanh \alpha_2}{1 + \tanh \alpha_1 \tanh \alpha_2}$ and by defining $\upsilon = \tanh \alpha$, the velocity υ is transformed to rapidity α which is analogous to a speed in rapidity space. For the range, $0 \le \upsilon \le 1$, the rapidity range is $0 \le \alpha \le \infty$. The rapidity α_3 of frame

3 relative to 1 is the sum of the rapidity α_1 of frame 2 relative to 1 and the rapidity α_2 of frame 3 relative to 2. Therefore, $\alpha_3 = (\alpha_1 + \alpha_2)$ and for many collinear frames, the rapidity of the last frame is the sum of the rapidities.

The standard velocity addition formula is consistent with the conditions to satisfy relativistic requirements. Its predictions have been well applied in investigating relativistic observations giving results consistent with the mathematical analysis. The Fizeau's experiment, within error limits, is interpreted as providing the empirical support of the formula and represents the direct test for it. The experiment as a conclusive test of the formula has been subjected to critical reviews based on the validity of the underlying assumptions in its interpretation. A case for consideration [1] is the validity of neglecting the Doppler effect of light interacting with moving media in the analysis of the observed interference fringe effect. On the argument [2] that the experiment provides corroboration of the phase speed of light wave but not for light particle (photons), a non-interferometric Fizeau-type experiment capable of measuring photons in moving media has been presented that may serve as a crucial test of the velocity addition formula.

2. The velocity addition formula from Euclidean space.

On the assumption frames move in euclidean space, the conventional velocities υ of collinear frames are added arithmetically in classical physics. With the introduction of relativistic physics, it became evident this assumption is only valid to a good approximation for the case $\upsilon<<1$. We infer that to validly add velocities arithmetically corresponding with classical physics for its whole range, the velocity υ has to be transformed as moving in a euclidean space analogue.

This corresponds with the proper velocity V in the euclidean space (ES) diagram [3]. From this reasoning, it is deduced that the proper velocity V_3 of frame 3 relative to 1 is the sum of the proper velocities V_1 and V_2 of frame 2 relative to 1 and frame 3 relative to 2 respectively.

$$V_3 = V_1 + V_2 \quad …….. \; Eqn\ 2$$

Since proper velocity expressed as a function of orientation angle ϕ and spacetime angle θ is $V = tan\ \phi$ and $V = cot\ \theta$ respectively, *Eqn 2* re-expressed in terms of these real rotation angles is

$$tan\ \phi_3 = tan\ \phi_1 + tan\ \phi_2 \quad ….. \; Eqn\ 3$$

or alternatively as

$$cot\ \theta_3 = cot\ \theta_1 + cot\ \theta_2 \quad ….. \; Eqn\ 4$$

In euclidean space $\upsilon = \sin\phi$ (and $V = \upsilon/\sqrt{(1-\upsilon^2)} = \tan\phi$) and in rapidity space $\upsilon = \tanh\alpha$. In the ES diagram, the angular velocity parameters ϕ (and θ) represents a 4-rotation in euclidean space compared to the angular velocity parameter α which is analogous to speeds in rapidity space. This explains why the formula is not derived as the sum of ϕ [4]. With proper velocities V as additive compared to rapidities α as additive in the standard case, the derived formula is inconsistent with the standard formula.

From *Eqn 3*, $\phi_3 = \arctan[\tan\phi_1 + \tan\phi_2]$. Since $\upsilon_1 = \sin\phi_1$; $\upsilon_2 = \sin\phi_2$ and $\upsilon_3 = \sin\phi_3$, substituting and re-arranging yields, $\upsilon_3 = \sin\{\arctan[\tan(\arcsin\upsilon_1) + \tan(\arcsin\upsilon_2)]\}$. This is identical to the alternative formula offered [5] based purely as a mathematical exercise.[1]

The proper velocity $V = \tan\phi$ $(= \cot\theta)$ of a frame is equal to the gradient g of its corresponding slope in the ES diagram. *Fig 1* shows the slopes in the circular representation [2] of the ES diagram.

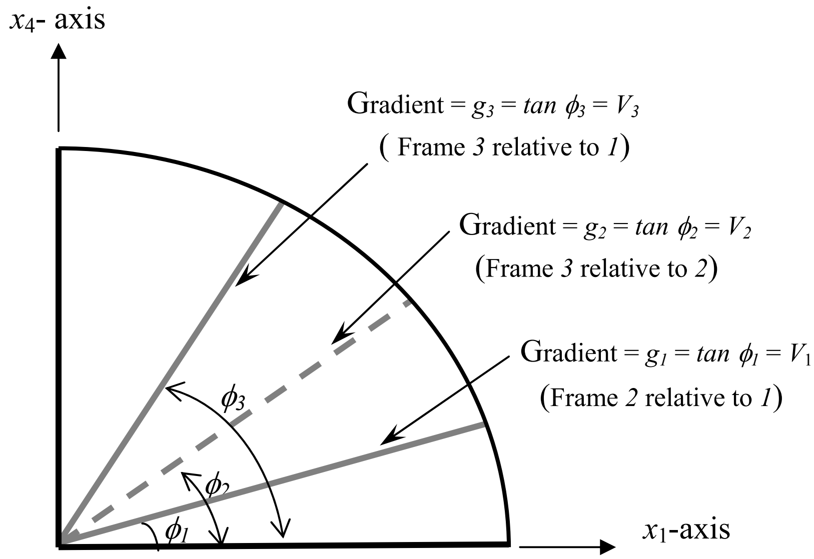

Figure 1: The gradients of collinear frames *1, 2* & *3* slopes in the ES diagram relative to *1*.

The proper velocity of frame *2* relative to *1*, V_1, is gradient , g_1 and of frame *3* relative to *2*, V_2 , is g_2 (in dotted lines since inserted into ES diagram relative to frame *1*) and of frame *3* relative to *1*, V_3 , is g_3 . From *Eqn 2*,

$$g_3 = g_1 + g_2 \quad \ldots\ldots \; Eqn\ 5$$

[1] For comparison the standard formula is, $\upsilon_3 = \tanh[\operatorname{arctanh}(\upsilon_1) + \operatorname{arctanh}(\upsilon_2)]$.

[2] In the ES diagram the magnitude of the proper velocity vector $\boldsymbol{V}$ given by the inclined lines length varies. Since our interest is only on its gradients, for convenience we use the circular representation.

For many collinear frames, we adopt a convention where υ_{ij} and V_{ij} are the conventional and proper velocities of frame *j* relative to frame *i*. Using this convention, from *Eqn 2*, the proper velocity of frame *4* relative to *1*, V_{14} , is equal to the sum of the proper velocities of frame *3* relative to *1*, V_{13} , and frame *4* relative to *3*, V_{34} .

$$V_{14} = V_{13} + V_{34} \quad \ldots\ldots\ldots \; Eqn\ 6$$

Substituting V_{13} from *Eqn 2* into *Eqn 6*,

$$V_{14} = V_{12} + V_{23} + V_{34}$$

In general the proper velocity of the n^{th} frame relative to frame *1* is

$$V_{1n} = V_{12} + V_{23} + V_{34} + \ldots\ldots + V_{(n-1)\,n}$$

Expressed in terms of ϕ is

$$tan\ \phi_{1n} = tan\ \phi_{12} + tan\ \phi_{23} + tan\ \phi_{34} + \ldots.. + tan\ \phi_{(n-1)\,n}$$

or in terms of θ is

$$cot\ \theta_{1n} = cot\ \theta_{12} + cot\ \theta_{23} + cot\ \theta_{34} + \ldots\ldots + cot\ \theta_{(n-1)n}$$

or in terms of *g* is

$$g_{1n} = g_{12} + g_{23} + g_{34} + \ldots + g_{(n-1)\,n}$$

3. The velocity transformation in euclidean space compared with rapidity space.

By transforming the conventional velocity υ into proper velocity V through the relationship $V = \dfrac{\upsilon}{\sqrt{1-\upsilon^2}}$, collinear frames are studied as moving in euclidean space compared with as moving in rapidity space at rapidity α through the relationship $\alpha = tanh^{-1}\,\upsilon$.

For the υ range $0 \le \upsilon \le 1$, V range is $0 \le V \le \infty$ and α range is $0 \le \alpha \le \infty$. Expressing υ in terms of V and α, $\upsilon = \dfrac{V}{\sqrt{1+V^2}}$ and $\upsilon = tanh\ \alpha$ respectively. *Fig 2* shows the velocity transformation for euclidean space with rapidity space graph superimposed on it.

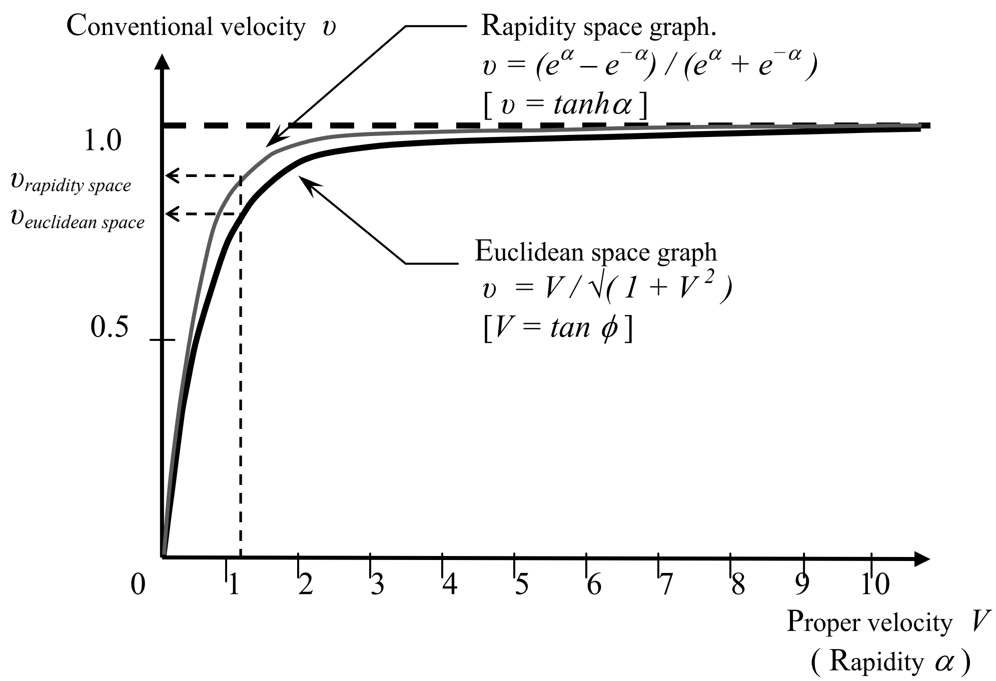

Figure 2 : Graph of conventional velocity υ vs proper velocity V in euclidean space compared with rapidity α in rapidity space.

From *Fig 2* , when $\upsilon<<1$, both the graphs are almost linear with $\upsilon \approx V \approx \alpha$ and the velocity transformation relationship reduces to the arithmetic sum of conventional velocities. As υ increases both graphs bends towards the horizontal $\upsilon = 1$ line. As υ approaches 1 ($\upsilon \rightarrow 1$), $V \rightarrow \infty$, $\alpha \rightarrow \infty$. This translates as the transformation relationship of the sum of all proper velocities or rapidities approaches a limiting value of $\upsilon = 1$ for both cases. This feature ensures if $\upsilon = 1$ for any frame, the last frame's velocity υ is also 1. Thus the characteristic of both graphs are identical and fulfills the conditions to satisfy the basic relativistic requirements for observations between collinear frames.

Adopting receding velocity as positive (+*ve*) and approaching velocity as negative (–*ve*), the euclidean space and rapidity space graphs in *Fig 2* are both symmetrical for these two ranges. The velocity υ range for the approaching motion case is $-1< \upsilon <0$ and its corresponding ϕ range is $-\frac{\pi}{2} < \phi < 0$. Since for this range of ϕ values, $sin\ (-\ \phi\ ) = -\ sin\ \phi$ and $tan\ (-\ \phi) = -\ tan\ \phi$ thus with this convention only the quadrant circle is needed for both directions of motion. If in terms of θ, then $V = cot\ \theta$ and $\upsilon = cos\ \theta$ with the results consistent with the above.

Fig 3 shows the percentage deviation of υ values in euclidean space compared to rapidity space graph. For the same proper velocity and rapidity, the corresponding conventional velocity υ are $\upsilon_{euclidean\ space}$ and $\upsilon_{rapidity\ space}$ respectively. Although V and α ranges to infinity the graph only shows to *10* units because the deviation is very small for higher values. The maximum percentage difference of *8.094%* occurs when V and α is *1.43*.

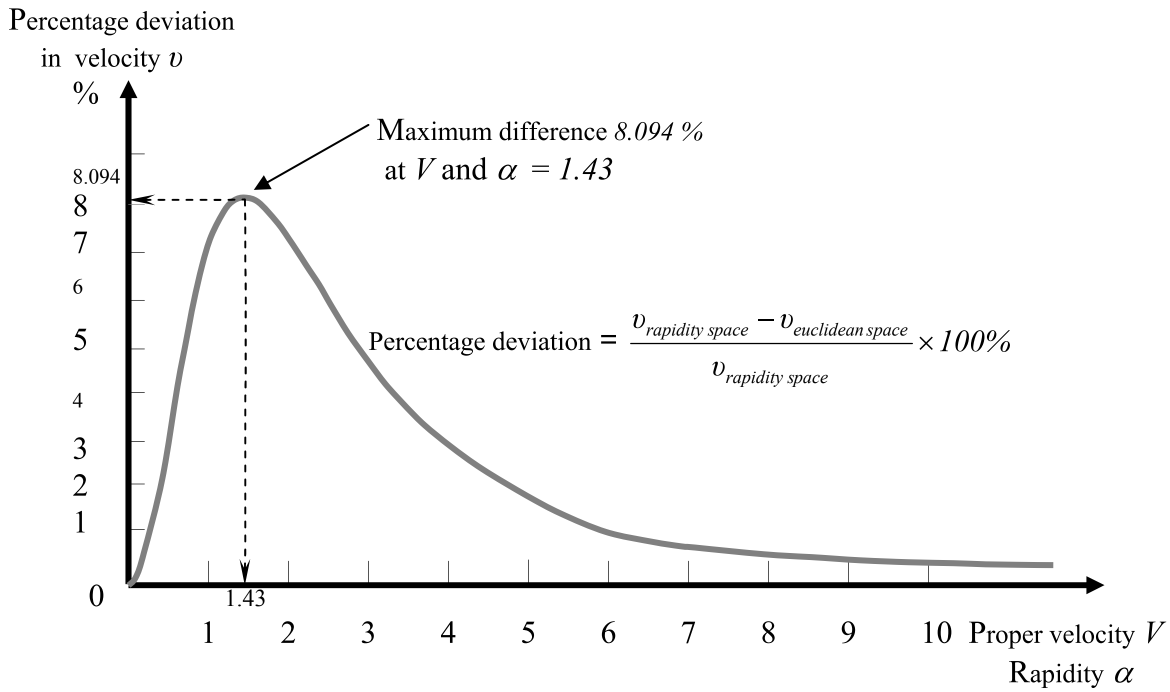

Figure 3: Graph of percentage deviation of velocity υ in euclidean space compared with rapidity space.

From $\upsilon = \tanh\alpha$ and $\upsilon = \sin\phi = \cos\theta$, applying trigonometric identities, we obtain $\upsilon = \tanh\alpha = \sin\phi = \cos\theta$; $V = \sinh\alpha = \tan\phi = \cot\theta$; $\cosh\alpha = \sec\phi = \operatorname{cosec}\theta$; $\operatorname{sech}\alpha = \cos\phi = \sin\theta$. From these relationships, the velocity addition formula and the relativistic space, time, momentum, energy and frequency equations can neatly be expressed interchangeably in terms of ϕ, ϑ or α. Similar to expressing $\alpha = \operatorname{arctanh}\upsilon$ in differential and integral form as $\frac{d\alpha}{d\upsilon} = \frac{1}{1-\upsilon^2}$ and $\alpha = \int \frac{d\upsilon}{1-\upsilon^2}$, we can express (a) $V = \tan\phi$ as $\frac{dV}{d\phi} = \sec^2\phi$ and $V = \int \sec^2\phi \, d\phi$ and (b) $V = \cot\vartheta$ as $\frac{dV}{d\theta} = -\operatorname{cosec}^2\vartheta$ and $V = \int -\operatorname{cosec}^2\theta \, d\vartheta$. From these, we obtain $\frac{d\alpha}{d\upsilon} = \frac{dV}{d\phi} = -\frac{dV}{d\theta} = \frac{1}{1-\upsilon^2} = \sec^2\phi = \operatorname{cosec}^2\vartheta = \cosh^2\alpha = 1 + V^2$.

In our formulation, the angles ϕ and θ are real, thus the identical characteristics between euclidean and rapidity space graphs and the neat mathematical relationships between ϕ, θ and α suggests rapidity α is not merely an abstract quantity. This corresponds with *Leblond* [6], "*The argument often heard that rapidity is a useful quantity, but a highly abstract one related only to 'physical' velocity through the formal expression (i.e. $\alpha = \operatorname{arctanh}\upsilon$), should be rebutted that, on the contrary, rapidity has an immediate physical significance .....*" .

4. Conlusion.

In investigating collinear frames, we interpreted that transforming the velocities in terms of a euclidean space analogue validates its arithmetical addition. As this corresponds with proper velocity V and not rapidity α, the deduced formula slightly deviates from the standard case. The formula expressed in terms of ϕ or θ in trigonometric form was also shown. The identical graphical characteristics and neat mathematical relationship between the speed parameters α , ϕ, θ, V and υ suggests an intimate link exist between rapidity space and euclidean space. Finally, it is imperative the interpretation problems in verifying the velocity addition formula be resolved to conclusively ascertain the formula that is excluded from consideration.

REFERENCES:

1. Renshaw, C: *The experiment of Fizeau as a test of relativistic simultaneity*. Available at http://renshaw.teleinc.com/papers/fizeau4b/fizeau4b.stm

2. Spavieri, G : *Locality and electromagnetic momentum in critical tests of special relativity*. Physical interpretations of relativity theory (PIRT) X, Imperial College, London. 2006.

3. Kanagaraj, S: *A Euclidean space analogue of a moving body*, Euclidean Relativity, Sept 2009. Available at http://www.euclideanrelativity.net

4. If derived based on the arithmetical addition of the angular velocity parameter ϕ in similarity with rapidity α where $\tanh \alpha_3 = \tanh (\alpha_1 + \alpha_2)$ in rapidity space, the relationship would then be $\sin \phi_3 = \sin (\phi_1 + \phi_2) = (\sin \phi_1 \cos \phi_2 + \cos \phi_1 \sin \phi_2)$
   Substituting, $\sin \phi = \upsilon/c$ and $\cos \phi = \sqrt{1- \upsilon^2 / c^2}$, the resulting formula would appear as :-

$$\upsilon_3 = \upsilon_1 \sqrt{1- \upsilon_2^{\,2} / c^2} + \upsilon_2 \sqrt{1- \upsilon_1^{\,2} / c^2}$$

5. Mathpages, *Relativistic speed composition formula*, Feb 2005. Available at http://www.mathpages.com/home/kmath494.htm

6. J-M Levy-Leblond, *Speeds*, Am J Phys, 48(5), 345-347, 1980.

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